GHZ States, Almost-Complex Structure and Yang–Baxter Equation (I)

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Abstract

Recent study suggests that there are natural connections between quantum information theory and the Yang–Baxter equation. In this paper, in terms of the generalized almost-complex structure and with the help of its algebra, we define the generalized Bell matrix to yield all the GHZ states from the product base, prove it to form a unitary braid representation and present a new type of solution of the quantum Yang–Baxter equation. We also study Yang-Baxterization, Hamiltonian, projectors, diagonalization, noncommutative geometry, quantum algebra and FRT dual algebra associated with this generalized Bell matrix.

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1 Introduction

Recently, a series of papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] have suggested there are natural and deep connections between quantum information theory [11] and the Yang–Baxter equation (YBE) [12, 13]. Unitary solutions of the braided YBE (i.e., the braid group relation) [1, 2] as well as unitary solutions of the quantum Yang–Baxter equation (QYBE) [3, 4] can be often identified with universal quantum gates [14]. Yang–Baxterization [15] is exploited to set up the Schrödinger equation determining the unitary evolution of a unitary braid gate [3, 4]. Furthermore, the Werner state [16] is viewed as a rational solution of the QYBE and the isotropic state [17] with a specific parameter forms a braid representation, see [7, 8]. More interestingly, the Temperley–Lieb algebra [18] deriving a braid representation in the state model for knot theory [19] is found to present a suitable mathematical framework for a unified description of various kinds of quantum teleportation phenomena [20], see [9, 10].

The present paper is a further extension of the previous published research work [3, 4, 6] in which the Bell matrix has been recognized to form a unitary braid representation and generate all the Bell states from the product base. In this paper, a unitary braid representation also called the Bell matrix for convenience is defined to create all the Greenberger-Horne-Zeilinger states (GHZ states) from the product base. The GHZ states are maximally multipartite entangled states (a natural generalization of the Bell states) and play important roles in the study of quantum information phenomena [21, 22, 23]. More importantly, this Bell matrix has a form in terms of the almost-complex structure which is fundamental for complex and Kähler geometry and symplectic geometry. Therefore, our paper is building heuristic connections among quantum information theory, the Yang–Baxter equation and differential geometry.

We hereby summarize our main result which is new to our knowledge.

1. We define the Bell matrix to produce all the GHZ states from the product base, prove it to be a unitary braid representation, and derive the Hamiltonian to determine the unitary evolution of the GHZ states.

2. We recognize the almost-complex structure in the formulation of the Bell matrix as well as its algebra in the proof for the Bell matrix satisfying the braided YBE, and exploit it to represent a new type of the solution of the QYBE.

3. We study topics associated with the generalized Bell matrix which include Yang–Baxterization, diagonalization, noncommutative geometry, quantum algebra via the RTT relation and standard FRT procedure [24, 25].

As the first paper in this research project, for simplicity, the present manuscript only focuses on the generalized Bell matrix of the type $2^{2n} \times 2^{2n}$ corresponding to the GHZ states of an even number of objects, while our result on the generalized Bell matrix of the type $2^{2n+1} \times 2^{2n+1}$ is collected [26].

The plan of this paper is organized as follows. Section 2 sketches the definition of the GHZ states and represent the Bell matrix in terms of the almost complex structure. Section 3 introduces the generalized Bell matrix and show both an algebra and a interesting type of solution of the QYBE in terms of the generalized almost-complex structure. Sections 4 and 5 briefly deal with various topics about the generalized Bell matrix: projectors, diagonalization, noncommutative geometry, quantum algebra and FRT dual algebra. Last section concludes with worthwhile problems for further research.

2 GHZ states, Bell matrix and Hamiltonian

This section is devised to set up a simplest example to be appreciated by readers mostly interested in quantum information and physics, and it explains how to observe the Bell matrix from the formulation of the GHZ states (as well as the almost-complex structure from the Bell matrix) and how to obtain Hamiltonians to determine the unitary evolution of the GHZ states.

2.1 GHZ states, Bell matrix and almost-complex structure

In the 2^N-dimensional Hilbert space with the base denoted by the Dirac kets $|m_1, m_2, \cdots, m_N\rangle$, $m_1, \cdots, m_N = \pm\frac{1}{2}$, there are 2^N linearly independent GHZ states of N-objects having the form

$$\frac{1}{\sqrt{2}}(|m_1, m_2, \cdots, m_N\rangle \pm |-m_1, -m_2, \cdots, -m_N\rangle) \tag{1}$$

which are maximally entangled states in quantum information theory [11]. In this paper, all the GHZ states are found to be generated by the Bell matrix acting on the chosen product base,

$$|\Phi_k\rangle = |m_1, m_2, \cdots, m_N\rangle, \quad |\Phi_{2^N-k+1}\rangle = |-m_1, -m_2, \cdots, -m_N\rangle, \tag{2}$$

where $1 \leq k \leq 2^{N-1}$. One can take a notation similar to [27, 28],

$$k[m_1, \cdots, m_N] = 2^{N-1} + \frac{1}{2} - \sum_{i=1}^{N} 2^{N-i}\ m_i \tag{3}$$

which has the result at $N = 2$, for example,

$$k[\frac{1}{2}, \frac{1}{2}] = 1,\ k[\frac{1}{2}, -\frac{1}{2}] = 2,\ k[-\frac{1}{2}, \frac{1}{2}] = 3,\ k[-\frac{1}{2}, -\frac{1}{2}] = 4, \tag{4}$$

assigned to label the GHZ states of two objects (the well known Bell states).

The 4×4 Bell matrix B_4 acts on the product base $|\frac{1}{2}\frac{1}{2}\rangle$, $|\frac{1}{2}\frac{-1}{2}\rangle$ and $|\frac{-1}{2}\frac{1}{2}\rangle$, $|\frac{-1}{2}\frac{-1}{2}\rangle$ to yield the Bell states, and it has a known form [1, 2, 3, 4, 6],

$$B_4 = (B_{kn,\,lm})_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad k, n, l, m = \frac{1}{2}, -\frac{1}{2}, \tag{5}$$

and the 8×8 Bell matrix B_8 given by

$$
B_8 \equiv (B_{\alpha l,\beta m})_8 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
\alpha, \beta = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \; l, m = \frac{1}{2}, -\frac{1}{2} \tag{6}
$$

creates the GHZ states of three objects by acting on $|\Phi_k\rangle$, $1 \leq k \leq 8$.

The $2^N \times 2^N$ Bell matrix generating the GHZ states of N-objects from the product base $|\Phi_k\rangle$, $1 \leq k \leq 2^N$, has a form in terms of the almost-complex structure [3] denoted by M,

$$
B = \frac{1}{\sqrt{2}}(1\!\!1 + M), \qquad B_{ij,kl} \equiv B_{ij}^{kl} = \frac{1}{\sqrt{2}}(\delta_i^k \delta_j^l + M_{ij}^{kl}) \tag{7}
$$

where $1\!\!1$ denotes the identity matrix, the lower index of B_{2^N} is omitted for convenience, δ_i^j is the Kronecker function of two variables i, j, which is 1 if $i = j$ and 0 otherwise, and the almost-complex structure M has the component formalism using the step function $\epsilon(i)$,

$$
M_{ij,kl} \equiv M_{ij}^{kl} = \epsilon(i)\delta_i^{-k}\delta_j^{-l}, \qquad \epsilon(i) = 1, i \geq 0; \;\; \epsilon(i) = -1, i < 0, \tag{8}
$$

which satisfies $M^2 = -1\!\!1$. In terms of the tensor product of the Pauli matrices, the Bell matrix B and the almost complex structure M for N-objects have the forms given by

$$
B = e^{\frac{\pi}{4}M}, \; M = \sqrt{-1}\sigma_y \otimes (\sigma_x)^{\otimes(N-1)}, \; (\sigma_x)^{\otimes(N-1)} = \underbrace{\sigma_x \otimes \cdots \otimes \sigma_x}_{N-1}. \tag{9}
$$

Note that there exist other interesting matrices related to the GHZ states, for example, one can have matrix entries $\epsilon(i)B_{ij,kl}$ for a new matrix. But so far as the authors know, only the Bell matrix is found to form a unitary braid representation.

2.2 Yang–Baxterization and Hamiltonian

The Bell matrix B satisfies the following characteristic equation given by

$$
(B - \frac{1+\sqrt{-1}}{\sqrt{2}}1\!\!1)(B - \frac{1-\sqrt{-1}}{\sqrt{2}}1\!\!1) = 0 \tag{10}
$$

[3] The almost-complex structure is usually denoted by the symbol J in the literature and it is a linear map from a real vector space to itself satisfying $J^2 = -1$. More details on geometry underlying what we are presenting here will be discussed elsewhere.

which suggests it having two distinct eigenvalues $\frac{1\pm\sqrt{-1}}{\sqrt{2}}$. Using Yang–Baxterization[4], a solution of the QYBE with the Bell matrix as its asymptotic limit, is obtained to be

$$\check{R}(x) = B + xB^{-1} = \frac{1}{\sqrt{2}}(1+x)1\!\!1 + \frac{1}{\sqrt{2}}(1-x)M. \tag{11}$$

As this solution $\check{R}(x)$ is required to be unitary, it needs a normalization factor ρ with a real spectral parameter x,

$$B(x) = \rho^{-\frac{1}{2}}\check{R}(x), \qquad \rho = 1+x^2,\ x \in \mathbb{R}. \tag{12}$$

As the real spectral parameter x plays the role of the time variable, the Schrödinger equation describing the unitary evolution of a state ϕ (independent of x) determined by the $B(x)$ matrix, i.e., $\psi(x) = B(x)\phi$, has the form

$$\sqrt{-1}\frac{\partial}{\partial x}\psi(x) = H(x)\psi(x), \qquad H(x) \equiv \sqrt{-1}\frac{\partial B(x)}{\partial x}B^{-1}(x), \tag{13}$$

where the time-dependent Hamiltonian $H(x)$ is given by

$$H(x) = \sqrt{-1}\frac{\partial}{\partial x}(\rho^{-\frac{1}{2}}\check{R}(x))(\rho^{-\frac{1}{2}}\check{R}(x))^{-1} = -\sqrt{-1}\rho^{-1}M. \tag{14}$$

To construct the time-independent Hamiltonian, a new time variable θ instead of the spectral parameter x is introduced in the way

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}, \qquad \sin\theta = \frac{x}{\sqrt{1+x^2}}, \tag{15}$$

so that the Bell matrix $B(x)$ has a new formulation as a function of θ,

$$B(\theta) = \cos\theta B + \sin\theta B^{-1} = e^{(\frac{\pi}{4}-\theta)M}, \tag{16}$$

and hence the Schrödinger equation for the time evolution of $\psi(\theta) = B(\theta)\phi$ has the form

$$\sqrt{-1}\frac{\partial}{\partial\theta}\psi(\theta) = H\psi(\theta), \qquad H \equiv \sqrt{-1}\frac{\partial B(\theta)}{\partial\theta}B^{-1}(\theta) = -\sqrt{-1}M, \tag{17}$$

where the time-independent Hamiltonian [5] H is hermitian due to the anti-hermitian of the almost-complex structure, i.e., $M^\dagger = -M$, and the unitary evolution operator $U(\theta)$ has the form $U(\theta) = e^{-M\theta}$.

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[4]See [4] or Subsection 3.1 and Subsection 4.1 for the detail. Here Yang–Baxterization is applied to the Bell matrix of the type $2^{2n}\times 2^{2n}$, while Yang–Baxterization of the Bell matrix of the type $2^{2n+1}\times 2^{2n+1}$ is rather subtle to be presented [26].

[5]The Hamiltonian used in our previous published papers [3, 4] has an additional numerical factor $\frac{1}{2}$ compared to the time-independent Hamiltonian (17). This factor $\frac{1}{2}$ is very important when we recognize the action of the four dimensional unitary evolution operator $\exp\frac{1}{2}\theta M$ on the product base to be equivalent to a product of two unitary rotations of Wigner functions for the Bell states $|\frac{1}{2}\frac{1}{2}\rangle \pm |\frac{-1}{2}\frac{-1}{2}\rangle$ and $|\frac{1}{2}\frac{-1}{2}\rangle \pm |\frac{-1}{2}\frac{1}{2}\rangle$, respectively. Note that no boundary conditions have been imposed on the Schrödinger equations (13) and (17).

3 Generalized Bell matrix and YBE

This section proves the generalized[6] Bell matrix $\widetilde{B}$ of the type $2^{2n} \times 2^{2n}$ to form a unitary braid representation with the help of the algebra generated by the generalized almost-complex structure $\widetilde{M}$, and presents an interesting type of solution of the QYBE in terms of $\widetilde{M}$ which may be not well noticed before in the literature.

3.1 YBE and Yang–Baxterization

In this paper, the braid group representation σ-matrix and the QYBE solution $\check{R}(x)$-matrix are $d^2 \times d^2$ matrices acting on $V \otimes V$ where V is a d-dimensional complex vector space. As σ and $\check{R}$ act on the tensor product $V_i \otimes V_{i+1}$, they are denoted by σ_i and $\check{R}_i$, respectively.

The generators σ_i of the braid group B_n satisfy the algebraic relation called the braid group relation,

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad 1 \le i \le n-1, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, \qquad |i-j| > 1. \end{aligned} \tag{18}$$

while the quantum Yang–Baxter equation (QYBE) has the form

$$\check{R}_i(x)\, \check{R}_{i+1}(xy)\, \check{R}_i(y) = \check{R}_{i+1}(y)\, \check{R}_i(xy)\, \check{R}_{i+1}(x) \tag{19}$$

with the spectral parameters x and y. In addition, the component formalism the QYBE (or the braid group relation) can be shown in terms of matrix entries,

$$\check{R}(x)_{i_1 j_1}^{i' j'} \check{R}(xy)_{j' k_1}^{k' k_2} \check{R}(y)_{i' k'}^{i_2 j_2} = \check{R}(y)_{j_1 k_1}^{j' k'} \check{R}(xy)_{i_1 j'}^{i_2 i'} \check{R}(x)_{i' k'}^{j_2 k_2}. \tag{20}$$

In view of the fact that $\check{R}(x = 0)$ forms a braid representation, the braid group relation is also called the braided YBE. Concerning relations between braid representations and x-dependent solutions of the QYBE (19), the procedure of constructing the $\check{R}(x)$-matrix from a given braid representation σ-matrix is called Baxterization [15] or Yang–Baxterization. For a braid representation σ with two distinct eigenvalues λ_1 and λ_2, the corresponding $\check{R}(x)$-matrix obtained via Yang–Baxterization has the form

$$\check{R}(x) = \sigma + x \lambda_1 \lambda_2 \sigma^{-1} \tag{21}$$

which has been exploited in Subsection 2.2, see (11).

3.2 Unitary generalized Bell matrix as a solution of YBE

The generalized Bell matrix $\widetilde{B}$ has the form in terms of the generalized almost-complex structure $\widetilde{M}$ with deformation parameters q_{ij},

$$\widetilde{B}_{ij}^{kl} = \frac{1}{\sqrt{2}} (\delta_i^k \delta_j^l + \widetilde{M}_{ij}^{kl}), \qquad \widetilde{M}_{ij}^{kl} = \epsilon(i) q_{ij} \delta_i^{-k} \delta_j^{-l}, \tag{22}$$

[6]Here "generalized" means that the object has deformation parameters.

where $q_{ij}q_{-i-j} = 1$ is required for $\widetilde{M}^2 = -1\!\!1$ and the step function $\epsilon(i)$ has the properties given by

$$\epsilon(i)\epsilon(i) = 1,\ \epsilon(i)\epsilon(-i) = -1,\ \epsilon(i) = \pm 1, \tag{23}$$

Let $\widetilde{B}$ be labeled by familiar indices by the angular momentum theory in quantum mechanics,

$$(\widetilde{B}^{J_1 J_2})^{b\nu}_{\mu a},\ \mu,\nu = J_1, J_1-1,\cdots,-J_1,\ a,b = J_2, J_2-1,\cdots,-J_2, \tag{24}$$

where $\widetilde{B}^{JJ}$ denotes the generalized Bell matrix $\widetilde{B}$ associated with the GHZ states of an even number of objects, for example,

$$\widetilde{B}_4 = \widetilde{B}^{\frac{1}{2}\frac{1}{2}},\ \widetilde{B}_{16} = \widetilde{B}^{\frac{3}{2}\frac{3}{2}},\ \widetilde{B}_{64} = \widetilde{B}^{\frac{7}{2}\frac{7}{2}}, \tag{25}$$

but the same type of generalized Bell matrix may be labeled differently, for example, both $\widetilde{B}^{\frac{1}{2}\frac{3}{2}}$ and $\widetilde{B}^{\frac{3}{2}\frac{1}{2}}$ are the type of $\widetilde{B}_8$.

In the following, we focus on the generalized Bell matrix of the type $\widetilde{B}^{JJ}$ denoted by $\widetilde{B}$, while we submit our result on the generalized Bell matrix of the type $\widetilde{B}^{J_1 J_2}$, $J_1 \neq J_2$ to [26].

In the proof for $\widetilde{B}^{JJ}$ forming a braid representation (18) in terms of its component formalism (22), deformation parameters q_{ij} are found to satisfy equations,

$$\begin{aligned} &q_{i_1 j_1} q_{-i_1-j_1} = q_{j_1 k_1} q_{-j_1-k_1}, && i_1, j_1, k_1 = J, J-1,\cdots,-J, \\ &q_{j_1 k_1} = q_{i_1 j_1} q_{-j_1 k_1} q_{-i_1 j_1}, && q_{i_1 j_1} = q_{j_1 k_1} q_{i_1 - j_1} q_{j_1 - k_1}, \end{aligned} \tag{26}$$

where no summation is imposed between same lower indices and which can be simplified by $q_{i_1 j_1} q_{-i_1-j_1} = 1$. Furthermore, the unitarity of $\widetilde{B}$ leads to a constraint on the generalized almost-complex structure $\widetilde{M}$, namely,

$$\widetilde{M}^\dagger \equiv \widetilde{M}^{*T} = \widetilde{M}^{-1} = -\widetilde{M} \Rightarrow q^*_{ij} q_{ij} = 1, \tag{27}$$

where the symbol $*$ denotes the complex conjugation and the symbol T denotes the transpose operation.

As J is a half-integer, we obtain solutions for equations (26) and (27) in terms of independent $(J+\frac{1}{2})$ number of angle parameters φ_J, φ_{J-1}, $\cdots$, $\varphi_{\frac{1}{2}}$,

$$q_{lm} = e^{i\frac{\varphi_l + \varphi_m}{2}},\ \varphi_{-l} = -\varphi_l, \qquad 0 \le l \le J, \tag{28}$$

where the method of separation of variables has been used since one can choose $q_l = e^{i\varphi_l}$ and then $q_{lm} = q_l q_m$.

For example, deformation parameters in the unitary generalized Bell matrix $\widetilde{B}^{\frac{1}{2}\frac{1}{2}}$ are calculated to be

$$q_{\frac{1}{2}\frac{1}{2}} = e^{i\varphi},\ q_{-\frac{1}{2}-\frac{1}{2}} = e^{-i\varphi},\ q_{\frac{1}{2}-\frac{1}{2}} = q_{-\frac{1}{2}\frac{1}{2}} = 1, \tag{29}$$

which are the same as those presented [3, 4, 6], and deformation parameters of the generalized Bell matrix $\widetilde{B}^{\frac{3}{2}\frac{3}{2}}$ have the form,

$$
\begin{aligned}
&q_{\frac{3}{2}\frac{3}{2}} = e^{i\varphi_1},\ q_{\frac{3}{2}\frac{1}{2}} = e^{i\frac{\varphi_1+\varphi_2}{2}},\ q_{\frac{3}{2}-\frac{1}{2}} = e^{i\frac{\varphi_1-\varphi_2}{2}},\ q_{\frac{3}{2}-\frac{3}{2}} = 1, \\
&q_{\frac{1}{2}\frac{3}{2}} = e^{i\frac{\varphi_1+\varphi_2}{2}},\ q_{\frac{1}{2}\frac{1}{2}} = e^{i\varphi_2},\ q_{\frac{1}{2}-\frac{1}{2}} = 1,\ q_{\frac{1}{2}-\frac{3}{2}} = e^{i\frac{\varphi_2-\varphi_1}{2}}.
\end{aligned}
\tag{30}
$$

In the 2^{2n}-dimensional[7] vector space, the generalized almost-complex structure $\widetilde{M}$ is found in this paper to satisfy algebraic relations,

$$
\begin{aligned}
&\widetilde{M}^2 = -1\!\!1, \qquad \widetilde{M}_{i\pm1}\widetilde{M}_i = -\widetilde{M}_i\widetilde{M}_{i\pm1}, \\
&\widetilde{M}_i\widetilde{M}_j = \widetilde{M}_j\widetilde{M}_i, \qquad |i-j| \geq 2,\ i,j \in \mathbb{N},
\end{aligned}
\tag{31}
$$

which defines an algebra obviously different from the Temperley–Lieb algebra [18] or the symmetric group algebra and where deformation parameters q_{ij} satisfy

$$
q_{ij}q_{-i-j} = 1, \qquad q_{ij}q_{-ij} = q_{jk}q_{j-k}. \tag{32}
$$

With the help of this algebra (31), the generalized Bell matrix $\widetilde{B}$ can be easily proved to satisfy the braided YBE (18) in the way

$$
\widetilde{B}_i\widetilde{B}_{i+1}\widetilde{B}_i = 2\widetilde{M}_i + 2\widetilde{M}_{i+1} + \widetilde{M}_i\widetilde{M}_{i+1} + \widetilde{M}_{i+1}\widetilde{M}_i = \widetilde{B}_{i+1}\widetilde{B}_i\widetilde{B}_{i+1}. \tag{33}
$$

Additionally, the generalized almost-complex structure $\widetilde{M}$ and the permutation operator P satisfy the following algebraic relation

$$
P_i\widetilde{M}_{i+1}P_i = P_{i+1}\widetilde{M}_iP_i, \qquad P = \sum_{ij} |ij\rangle\langle ji|, \tag{34}
$$

which is underlying algebraic relations of the virtual braid group, i.e., the braid $\widetilde{B}$ and permutation P forming a unitary virtual braid representation, see [7, 8].

3.3 New type of solution of QYBE via parameterization

Similar to the formalism of the rational solution of the QYBE (19),

$$
\check{R}_{rational}(u) = 1\!\!1 + uP, \qquad P^2 = 1\!\!1 \tag{35}
$$

where P is a permutation matrix, we obtain a solution of the QYBE in terms of the generalized almost-complex structure,

$$
\widetilde{\check{R}}(u) = 1\!\!1 + u\widetilde{M} \tag{36}
$$

satisfying the following equation of Yang–Baxter type,

$$
\check{R}_i(u)\,\check{R}_{i+1}\left(\frac{u+v}{1+uv}\right)\check{R}_i(v) = \check{R}_{i+1}(v)\,\check{R}_i\left(\frac{u+v}{1+uv}\right)\check{R}_{i+1}(u), \tag{37}
$$

[7]Here we have $2^{2n} = (2J+1)^2$, for example, $n=1, J=\frac{1}{2}$ and $n=2, J=\frac{3}{2}$, see (25).

which has been exploited [4] and where new spectral parameters u, v are related to original spectral parameters x, y in the way

$$u = \frac{1-x}{1+x}, \qquad v = \frac{1-y}{1+y}, \qquad \frac{1-xy}{1+xy} = \frac{u+v}{1+uv}. \tag{38}$$

Via a further parametrization of spectral parameters u, v in terms of angle variables Θ_1, Θ_2,

$$u = -\sqrt{-1}\tan\Theta_1,\ v = -\sqrt{-1}\tan\Theta_2,\ \frac{u+v}{1+uv} = -\sqrt{-1}\tan(\Theta_1+\Theta_2), \tag{39}$$

the modified Yang–Baxter equation (37) has the ordinary form

$$\check{R}_i(\Theta_1)\,\check{R}_{i+1}(\Theta_1+\Theta_2)\,\check{R}_i(\Theta_2) = \check{R}_{i+1}(\Theta_2)\,\check{R}_i(\Theta_1+\Theta_2)\,\check{R}_{i+1}(\Theta_1). \tag{40}$$

with the solution given by

$$\widetilde{\check{R}}(\Theta) = 1\!\!1 - \sqrt{-1}\tan\Theta\widetilde{M}, \qquad \text{or} \qquad \widetilde{\check{R}}(\Theta') = 1\!\!1 + \tanh\Theta'\widetilde{M}. \tag{41}$$

Note that physical models underlying this type of solution of QYBE in terms of the almost-complex structure will be discussed and submitted elsewhere.

4 Projectors, diagonalization and geometry

This section and the next one are aimed at introducing several selective topics directly using the generalized Bell matrix and the generalized almost-complex structure, for example, associated noncommutative geometry, quantum algebra and FRT dual algebra.

4.1 Projectors and Yang–Baxterization

In terms of $\widetilde{M}$, two projectors $\widetilde{P}_+$ and $\widetilde{P}_-$ are defined by

$$\widetilde{P}_+ = \frac{1}{2}(1+\sqrt{-1}\widetilde{M}), \qquad \widetilde{P}_- = \frac{1}{2}(1-\sqrt{-1}\widetilde{M}) \tag{42}$$

satisfying basic properties of two mutually orthogonal projectors,

$$\widetilde{P}_+ + \widetilde{P}_- = 1\!\!1,\ \widetilde{P}_\pm^2 = \widetilde{P}_\pm,\ \widetilde{P}_+\widetilde{P}_- = 0. \tag{43}$$

The generalized Bell matrix $\widetilde{B}$ has two distinct eigenvalues $e^{\pm i\frac{\pi}{4}}$ and it satisfies the same characteristic equation as (10),

$$(\widetilde{B} - \lambda_- 1\!\!1)(\widetilde{B} - \lambda_+ 1\!\!1) = 0, \qquad \lambda_+ = e^{-i\frac{\pi}{4}},\ \lambda_- = e^{i\frac{\pi}{4}}. \tag{44}$$

With the projectors $\widetilde{P}_\pm$ and eigenvalues $\lambda_\pm$, the generalized Bell matrix and its inverse have the forms

$$\widetilde{B} = \lambda_+\widetilde{P}_+ + \lambda_-\widetilde{P}_-, \qquad \widetilde{B}^{-1} = \lambda_-\widetilde{P}_+ + \lambda_+\widetilde{P}_-. \tag{45}$$

Using Yang–Baxterization [4], the $\check{R}(x)$-matrix as a solution of the QYBE (19) has a form similar to (11),

$$\widetilde{\check{R}}(x) = (\lambda_+ + \lambda_- x)\widetilde{P}_+ + (\lambda_- + \lambda_+ x)\widetilde{P}_- = \widetilde{B} + x\widetilde{B}^{-1}, \tag{46}$$

and hence the corresponding Schrodinger equation also has a similar form to (13) or (17) except that the Hamiltonian is determined by $\widetilde{M}$ instead of M.

4.2 Diagonalization of the generalized Bell matrix

The diagonalization of the generalized Bell matrix $\widetilde{B}$ can be performed by a unitary matrix D via the following unitary transformation,

$$D\widetilde{B}D^\dagger = \frac{1}{\sqrt{2}} Diag(1+\sqrt{-1}, \cdots, 1-\sqrt{-1}) \tag{47}$$

where the diagonal matrix $Diag$ has the same number of matrix entries $1+\sqrt{-1}$ as $1-\sqrt{-1}$. Assume D to have the form by a Hermitian unitary matrix N,

$$D = \frac{1}{\sqrt{2}}(1\!\!1 + \sqrt{-1}N), \qquad N^\dagger = N, \qquad N^2 = 1\!\!1 \tag{48}$$

and then this matrix N is found to satisfy an additional condition,

$$N\widetilde{M} = -\widetilde{M}N = Diag(1, -1, \cdots, 1, -1), \tag{49}$$

where the diagonal matrix $Diag$ has the same number of matrix entries 1 as -1 but the ordering between 1 and -1 is not fixed.

After some algebra, one type of formalism of the matrix N is given by

$$N_{ij}^{kl} = f(i) q_{ij} \delta_i^{-k} \delta_j^{-l}, \qquad f(i)f(i) = 1,\ f(i) = f(-i) = f^*(i) \tag{50}$$

where q_{ij} are the same as unitary deformation parameters q_{ij} in the generalized Bell matrix $\widetilde{B}$. This matrix N brings about the diagonalization form of $\widetilde{B}$,

$$(D\widetilde{B}D^\dagger)_{ij}^{mn} = \frac{1}{\sqrt{2}}(1 + \sqrt{-1} f(i)\epsilon(-i))\delta_i^m \delta_j^n. \tag{51}$$

in which setting $f(i) = \epsilon(-i)$, $i > 0$ and $f(i) = \epsilon(i)$, $i < 0$ leads to

$$D\widetilde{B}D^\dagger = \frac{1}{\sqrt{2}} Diag(\underbrace{1+\sqrt{-1}, \cdots, 1+\sqrt{-1}}_{2^{N-1}}, \underbrace{1-\sqrt{-1}, \cdots, 1-\sqrt{-1}}_{2^{N-1}}). \tag{52}$$

In the four dimensional case, for example, the Bell matrix B_4 is diagonalized in the way,

$$D_4 B_4 D_4^\dagger = \frac{1}{\sqrt{2}} Diag(1-\sqrt{-1}, 1+\sqrt{-1}, 1-\sqrt{-1}, 1+\sqrt{-1}),\ N_4 = -\sigma_y \otimes \sigma_y, \tag{53}$$

where a note is added that B_4 can be also diagonalized by unitary transformations of the Malkline matrix (or the magic matrix) [27, 28, 29] or the diagonaliser [30], and the following generalized Bell matrix $\widetilde{B}_4$ can be diagonalized with a given $N_{4,1}$,

$$\widetilde{B}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & q \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -q^{-1} & 0 & 0 & 1 \end{pmatrix}, \; N_{4,1} = \begin{pmatrix} 0 & 0 & 0 & -q \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \end{pmatrix},$$
$$D_{4,1}\widetilde{B}_4 D_{4,1} = \frac{1}{\sqrt{2}} Diag(1+\sqrt{-1}, 1+\sqrt{-1}, 1-\sqrt{-1}, 1-\sqrt{-1}). \tag{54}$$

As a remark, calculation for noncommutative geometry and quantum algebra associated with the generalized Bell matrix can be greatly simplified once the above diagonalization procedure is exploited.

4.3 Associated noncommutative geometry

With the help of the standard procedure of setting up associated noncommutative geometry with a given braid representation [31, 32], we denote coordinate operators X and differential operators ξ in the way

$$X^T = (x_1, x_2 \cdots, x_{2^N}), \qquad \xi^T = (\xi_1, \xi_2, \cdots, \xi_{2^N}) \tag{55}$$

and demand them to satisfy constraint equations,

$$\widetilde{P}_-(X \otimes X) = 0, \qquad \widetilde{P}_+(\xi \otimes \xi) = 0, \qquad X \otimes \xi = (\mu \widetilde{P}_+ - 1\!\!1)(\xi \otimes X) \tag{56}$$

where μ is a free parameter. These three equations can be chosen in the second way by exchanging $\widetilde{P}_+$ with $\widetilde{P}_-$ but this approach is omitted here for simplicity.

Hence noncommutative differential geometry generated by X and ξ is essentially determined by the following equations in terms of the generalized almost-complex structure $\widetilde{M}$,

$$X \otimes X = \sqrt{-1}\widetilde{M}(X \otimes X), \qquad \xi \otimes \xi = -\sqrt{-1}\widetilde{M}(\xi \otimes \xi),$$
$$X \otimes \xi = (\frac{\mu}{2} - 1)\xi \otimes X + \frac{\mu}{2}\sqrt{-1}\widetilde{M}(\xi \otimes X), \tag{57}$$

which have the following formalisms of component,

$$x_i x_j = \sqrt{-1}\epsilon(i) q_{ij} x_{-i} x_{-j}, \qquad \xi_i \xi_j = -\sqrt{-1}\epsilon(i) q_{ij} \xi_{-i}\xi_{-j},$$
$$x_i \xi_j = (\frac{\mu}{2} - 1)\xi_i x_j + \frac{\mu}{2}\sqrt{-1}\epsilon(i) q_{ij} \xi_{-i} x_{-j}. \tag{58}$$

with the significant geometry at $\mu = 2$. Note that noncommutative plane related to the Bell matrix B_4 has been briefly discussed [33].

5 Quantum algebra via the FRT procedure

For a given solution $\check{R}$ of the braided YBE (18), there is a standard procedure [24, 25] using the $\check{R}TT$ relation and $\check{R}LL$ relations to respectively define associated quantum algebra and FRT dual algebra. In this section, we sketch quantum algebra and FRT dual algebra specified by the $\widetilde{B}TT$ relation and $\widetilde{B}LL$ relations.

5.1 Quantum algebra using the $\widetilde{M}TT$ relation

In the well known $\check{R}TT$ relation: $\check{R}(T\otimes T) = (T\otimes T)\check{R}$, matrix entries of the T-matrix are assumed to be non-commutative operators. Here the $\check{R}$-matrix is the generalized Bell matrix $\widetilde{B}$, and the $\widetilde{B}TT$ relation is essentially determined by $\widetilde{M}TT$ relation,

$$\widetilde{B}(T\otimes T) = (T\otimes T)\widetilde{B} \Rightarrow \widetilde{M}(T\otimes T) = (T\otimes T)\widetilde{M}, \tag{59}$$

where $\widetilde{M}$ is a $2^{2n}\times 2^{2n}$ matrix and T is a $2^{2n-1}\times 2^{2n-1}$ matrix. In terms of matrix entries of $\widetilde{M}$, T and the convention $(A\otimes B)_{ij,kl} \equiv A_{ik}B_{jl}$, the $\widetilde{M}TT$ relation has the following component formalism,

$$\begin{aligned} & T_{i_1-i_2}T_{j_1-j_2} + \epsilon(i_1)\epsilon(i_2)q_{i_1j_1}q_{i_2j_2}T_{-i_1i_2}T_{-j_1j_2} = 0, \\ & i_1, i_2, j_1, j_2 = J, J-1, \cdots, -J. \end{aligned} \tag{60}$$

Note that this $\widetilde{M}TT$ relation (60) has eight simplified equations of component,

$$\begin{aligned} & T_{ii}T_{ii} = T_{-i-i}T_{-i-i}, && T_{ii}T_{-i-i} = T_{-i-i}T_{ii}, \\ & T_{i-i}T_{i-i} = -q_{ii}^2T_{-ii}T_{-ii}, && T_{i-i}T_{-ii} = -T_{-ii}T_{i-i}, \\ & T_{ii}T_{i-i} = q_{ii}T_{-i-i}T_{-ii}, && T_{ii}T_{-ii} = q_{-i-i}T_{-i-i}T_{i-i}, \\ & T_{i-i}T_{ii} = -q_{ii}T_{-ii}T_{-i-i}, && T_{i-i}T_{-i-i} = -q_{ii}T_{-ii}T_{ii} \end{aligned} \tag{61}$$

which determine the quantum algebra related to $\widetilde{B}_4$.

With the help of a new $\widetilde{T}$-matrix given by

$$\widetilde{T}_{ij} = \epsilon(i)T_{ij} + T_{-i-j}, \qquad \widetilde{T}_{-i-j} = -\epsilon(i)T_{-i-j} + T_{ij}, \tag{62}$$

where q_{ij} is chosen to be unit for convenience, the $\widetilde{M}TT$ relation (60) is replaced by the $\widetilde{M}\widetilde{T}\widetilde{T}$ relation having the algebraic relations,

$$\begin{aligned} & \widetilde{T}_{i_1-i_2}\widetilde{T}_{j_1-j_2} = -\widetilde{T}_{-i_1i_2}\widetilde{T}_{-j_1j_2}, && \text{if} \quad \epsilon(i_1)\epsilon(i_2) = 1,\ \epsilon(i_2)\epsilon(j_1) = 1, \\ & \widetilde{T}_{i_1-i_2}\widetilde{T}_{-j_1j_2} = \widetilde{T}_{-i_1i_2}\widetilde{T}_{j_1-j_2}, && \text{if} \quad \epsilon(i_1)\epsilon(i_2) = 1,\ \epsilon(i_2)\epsilon(j_1) = -1, \\ & \widetilde{T}_{i_1-i_2}\widetilde{T}_{j_1-j_2} = \widetilde{T}_{-i_1i_2}\widetilde{T}_{-j_1j_2}, && \text{if} \quad \epsilon(i_1)\epsilon(i_2) = -1,\ \epsilon(i_2)\epsilon(j_1) = 1, \\ & \widetilde{T}_{i_1-i_2}\widetilde{T}_{-j_1j_2} = -\widetilde{T}_{-i_1i_2}\widetilde{T}_{j_1-j_2}, && \text{if} \quad \epsilon(i_1)\epsilon(i_2) = -1,\ \epsilon(i_2)\epsilon(j_1) = -1, \end{aligned} \tag{63}$$

which leads to four simplest algebraic relations,

$$\widetilde{T}^2_{i-i} = 0,\ \widetilde{T}_{ii}\widetilde{T}_{-i-i} = 0,\ \widetilde{T}_{ii}\widetilde{T}_{-ii} = 0,\ \widetilde{T}_{i-i}\widetilde{T}_{ii} = 0. \tag{64}$$

5.2 Example: the quantum algebra from the $\widetilde{B}_4TT$ relation

The $\widetilde{B}_4$-matrix and T-matrix take the forms,

$$\widetilde{B}_4 = \begin{pmatrix} 1 & 0 & 0 & q \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -q^{-1} & 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}, \tag{65}$$

and the $\widetilde{B}_4TT$ relation leads to the quantum algebra generated by $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ satisfying algebraic relations,

$$\begin{aligned} \hat{a}\hat{a} &= \hat{d}\hat{d},\ \hat{a}\hat{b} = q\hat{d}\hat{c},\ \hat{b}\hat{b} = -q^2\hat{c}\hat{c},\ \hat{a}\hat{c} = q^{-1}\hat{d}\hat{b}, \\ \hat{a}\hat{d} &= \hat{d}\hat{a},\ \hat{b}\hat{a} = -q\hat{c}\hat{d},\ \hat{b}\hat{c} = -\hat{c}\hat{b},\ \hat{c}\hat{a} = -q^{-1}\hat{b}\hat{d}, \end{aligned} \tag{66}$$

where the deformation parameter q can be absorbed into the generator $\hat{c}$ by a rescaling transformation. With the new operators $\widetilde{\hat{a}}, \widetilde{\hat{b}}, \widetilde{\hat{c}}, \widetilde{\hat{d}}$ [34] specified by

$$\widetilde{\hat{a}} = \hat{a} + \hat{d},\ \widetilde{\hat{b}} = \hat{b} + \hat{c},\ \widetilde{\hat{c}} = \hat{b} - \hat{c},\ \widetilde{\hat{c}} = \hat{a} - \hat{d}, \tag{67}$$

the above algebraic relations have a very simplified formalism,

$$\widetilde{\hat{a}}\widetilde{\hat{d}} = \widetilde{\hat{d}}\widetilde{\hat{a}} = 0,\ \widetilde{\hat{b}}\widetilde{\hat{b}} = \widetilde{\hat{c}}\widetilde{\hat{c}} = 0,\ \widetilde{\hat{a}}\widetilde{\hat{c}} = \widetilde{\hat{d}}\widetilde{\hat{b}} = 0,\ \widetilde{\hat{b}}\widetilde{\hat{a}} = \widetilde{\hat{c}}\widetilde{\hat{d}} = 0. \tag{68}$$

Note that the quantum algebra from the B_4TT relation and its representation theory has been presented [34], while the same quantum algebra from the $\widetilde{B}_4TT$ relation, interesting algebraic structures underlying its representation and its natural connection to quantum information theory has been explored [6].

As a remark, quantum algebra obtained from $\widetilde{B}TT$ relation may be higher-dimensional representations of that algebra given by $\widetilde{B}_4TT$ relation, see [26].

5.3 FRT dual algebra using the $\widetilde{M}LL$ relations

The $\check{R}LL$ relations determining the FRT dual algebra can be derived from the generalized $\check{R}TT$ relation which relies on the spectral parameter,

$$\check{R}(xy^{-1})(L(x) \otimes L(y)) = (L(y) \otimes L(x))\check{R}(xy^{-1}). \tag{69}$$

Assume the $L(x)$-matrix to have a similar form to $\widetilde{B}(x)$,

$$L(x) = L^+ + x\,L^-, \qquad \widetilde{B}(x) = \widetilde{B} + x\widetilde{B}^{-1}, \tag{70}$$

and this leads to the $\widetilde{B}LL$ relations for the FRT dual algebra,

$$\widetilde{B}(L^\pm \otimes L^\pm) = (L^\pm \otimes L^\pm)\widetilde{B}, \qquad \widetilde{B}(L^+ \otimes L^-) = (L^- \otimes L^+)\widetilde{B}, \tag{71}$$

where matrix entries of $L^{\pm}$ are non-commutative operators. These $\widetilde{B}LL$ relations are found to be essentially $\widetilde{M}LL$ relations,

$$\begin{aligned}
&\widetilde{M}(L^{\pm}\otimes L^{\pm}) = (L^{\pm}\otimes L^{\pm})\widetilde{M},\\
&L^{+}\otimes L^{-} - L^{-}\otimes L^{+} + \widetilde{M}(L^{+}\otimes L^{-}) - (L^{-}\otimes L^{+})\widetilde{M} = 0,
\end{aligned} \tag{72}$$

which have the component formalisms,

$$\begin{aligned}
&L^{\pm}_{i_1 -i_2} L^{\pm}_{j_1 -j_2} + \epsilon(i_1)\epsilon(i_2) q_{i_1 j_1} q_{i_2 j_2} L^{\pm}_{-i_1 i_2} L^{\pm}_{-j_1 j_2} = 0,\\
&L^{+}_{i_1 i_2} L^{-}_{j_1 j_2} - L^{-}_{i_1 i_2} L^{+}_{j_1 j_2} + \epsilon(i_1) q_{i_1 j_1} L^{+}_{-i_1 i_2} L^{-}_{-j_1 j_2} + \epsilon(i_2) q_{-i_2 -j_2} L^{-}_{i_1 -i_2} L^{+}_{j_1 -j_2} = 0,\\
&i_1, i_2, j_1, j_2 = J, J-1, \cdots, -J,
\end{aligned} \tag{73}$$

Note that the FRT dual algebra for the Bell matrix B_4 has been given [30] and a quotient algebra of this FRT dual algebra with the quotient condition $L^{+}\otimes L^{-} = L^{-}\otimes L^{+}$ has been presented [6]. Also, in view of [30, 34, 6], further research is needed to construct representation theories and seek interesting algebraic structures underlying them for these quantum algebra and FRT dual algebra.

6 Concluding remarks and outlooks

This paper is motivated by the recent study [1, 2, 3, 4, 6], and it sheds a light on further research for unraveling deep connections among quantum information theory, Yang–Baxter equation and complex geometry. We find that the GHZ states can be yielded by the Bell matrix on the product base and prove that the generalized Bell matrix of the type $2^{2n}\times 2^{2n}$ forms a unitary braid representation with the help of the algebra generated by the almost-complex structure. The algebraic and diagrammatic proofs for the generalized Bell matrix of the type $2^{2n+1}\times 2^{2n+1}$ satisfying the braided YBE together with other interesting result will be submitted [26].

Besides what we have done in the present paper, there still remain many meaningful topics worthwhile to be explored. For example, almost-complex structure, classical YBE and symplectic geometry; construction of a universal R-matrix [35] in terms of the generators of the algebra from the $\widetilde{B}_4TT$ relation; Yangian, Yang–Baxter equation and quantum information; new quantum algebra obtained by exploiting methodologies for the Sklyanin algebra [36, 37] to the generalized Bell matrix. The most important thing (at least for the authors) is still to look for further connections among physics, quantum information and the YBE.

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References

[1] H.A. Dye, *Unitary Solutions to the Yang–Baxter Equation in Dimension Four*, Quant. Inf. Proc. **2** (2003) 117-150. Arxiv: quant-ph/0211050.

[2] L.H. Kauffman and S.J. Lomonaco Jr., *Braiding Operators are Universal Quantum Gates*, New J. Phys. **6** (2004) 134. Arxiv: quant-ph/0401090.

[3] Y. Zhang, L.H. Kauffman and M.L. Ge, *Universal Quantum Gate, Yang–Baxterization and Hamiltonian*. Int. J. Quant. Inform., Vol. 3, **4** (2005) 669-678. Arxiv: quant-ph/0412095.

[4] Y. Zhang, L.H. Kauffman and M.L. Ge, *Yang–Baxterizations, Universal Quantum Gates and Hamiltonians*. Quant. Inf. Proc. **4** (2005) 159-197. Arxiv: quant-ph/0502015.

[5] J. Franko, E.C. Rowell and Z.H. Wang, *Extraspecial 2-Groups and Images of Braid Group Representations*. J. Knot Theory Ramifications, **15** (2006) 413-428. Arxiv: math.RT/0503435.

[6] Y. Zhang, N. Jing and M.L. Ge, *Quantum Algebras Associated With Bell States*. Arxiv: math-ph/0610036.

[7] Y. Zhang, L.H. Kauffman and R.F. Werner, *Permutation and its Partial Transpose*. Arxiv: quant-ph/0606005. Accepted by *International Journal of Quantum Information* for publication.

[8] Y. Zhang, L.H. Kauffman and M.L. Ge, *Virtual Extension of Temperley–Lieb Algebra*. Arxiv: math-ph/0610052.

[9] Y. Zhang, *Teleportation, Braid Group and Temperley–Lieb Algebra*. J. Phys. **A:** Math. Gen. **39** (2006) 11599-11622. Arxiv: quant-ph/0610148.

[10] Y. Zhang, *Algebraic Structures Underlying Quantum Information Protocols*. Arxiv: quant-ph/0601050.

[11] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 1999).

[12] C.N. Yang, *Some Exact Results for the Many Body Problems in One Dimension with Repulsive Delta Function Interaction*, Phys. Rev. Lett. **19** (1967) 1312-1314.

[13] R.J. Baxter, *Partition Function of the Eight-Vertex Lattice Model*, Annals Phys. **70** (1972) 193-228.

[14] J.L. Brylinski and R. Brylinski, *Universal Quantum Gates*, in *Mathematics of Quantum Computation*, Chapman & Hall/CRC Press, Boca Raton, Florida, 2002 (edited by R. Brylinski and G. Chen).

[15] V.F.R. Jones, *Baxterization*, Int. J. Mod. Phys. A **6** (1991) 2035-2043.

[16] R.F. Werner, *Quantum States with Einstein-Podolsky-Rosen Correlations Admitting a Hidden-Variable Model*, Phys. Rev. A **4**0 (1989) 4277.

[17] M. Horodecki and P. Horodecki, *Reduction Criterion of Separability and Limits for A Class of Distillation Protoco*, Phys. Rev. A **59** (1999) 4206.

[18] H.N.V. Temperley and E.H. Lieb, *Relations between the 'Percolation' and 'Colouring' Problem and Other Graph-Theoretical Problems Associated with Regular Planar Lattices: Some Exact Results for the 'Percolation' Problem*, Proc. Roy. Soc. A **322** (1971) 251-280.

[19] L.H. Kauffman, *Knots and Physics* (World Scientific Publishers, 2002).

[20] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, *Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels*, Phys. Rev. Lett. **70** (1993) 1895-1899.

[21] D.M. Greenberger, M.A.Horne and A.Zeilinger, *Going beyond Bell's Theorem*, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos, pp. 73–76, (Kluwer Academic, Dordrecht, 1989).

[22] D.M. Greenberger, M.A. Horne, A. Shirnony, and A.Zeilinger, *Bell's Theorem Without Inequalities*, Am. J. Phys. **58** (1990) 1131-1143.

[23] D. Bouwmeester, J.-W. Pan, M. Daniell, H. Weinfurter and A. Zeilinger, *Observation of Three-Photon Greenberger-Horne-Zeilinger Entanglement*, Phys. Rev. Lett. **82**(1999) 1345-1349.

[24] N.Yu. Reshetikhin, L.A. Takhtadjian and L.D. Faddeev, *Quantization of Lie Groups and Lie Algebras*. Algebra i Analiz **1** (1989) 178-206. English transl.: Leningrad Math. J. 1 (1990) 193-225.

[25] L.D. Faddeev, *Lectures on Quantum Inverse Scattering Method*, in Nankai Lectures on Mathematical Physics, "Integrable Systems", pp. 23-70 (World Scientific, 1987).

[26] Y. Zhang, K. Xue and M.L. Ge, *GHZ States, Almost-Complex Structure and Yang–Baxter Equation (II)*, (in preparation).

[27] K. Fujii and T. Suzuki, *On the Magic Matrix by Makhlin and the B-C-H Formula in SO(4)*, ArXiv: quant-ph/0610009.

[28] K. Fujii, H. Oike and T. Suzuki, *More on the Isomorphism* $SU(2)\otimes SU(2) \cong SO(4)$, Arxiv: quant-ph/0608186.

[29] Y. Makhlin, *Nonlocal Properties of Two-Qubit Gates and Mixed States and Optimization of Quantum Computations*, Quant. Info. Proc. **1** (2002) 243. Arxiv: quant-ph/0002045.

[30] D. Arnaudon, A. Chakrabarti, V.K. Dobrev and S.G. Mihov, *Spectral Decomposition and Baxterisation of Exotic Bialgebras and Associated Noncommutative Geometries*, Int. J. Mod. Phys. **A18** (2003) 4201-4213. Arxiv: math.QA/0209321.

[31] J. Wess and B. Zumino, *Covariant Differential Calculus on the Quantum Hyperplane*, Nucl. Phys. (Proc. Suppl.) **18** (1990) 302.

[32] J. Madore, *An Introduction to Noncommutative Differential Geometry* (Cambridge University Press, 1999).

[33] D. Arnaudon, A. Chakrabarti, V.K. Dobrev and S.G. Mihov, *Exotic Bialgebra S03: Representations, Baxterisation and Applications*, to appear in Annales Henri Poincare. Arxiv: math.QA/0601708.

[34] D. Arnaudon, A. Chakrabarti, V.K. Dobrev and S.G. Mihov, *Duality and Representations for New Exotic Bialgebras*, J. Math. Phys. **43** (2002) 6238-6264. Arxiv: math.QA/0206053.

[35] C. Kassel, *Quantum Groups* (Spring-Verlag New York, 1995).

[36] E.K. Sklyanin, *Some Algebraic Structures Connected with the Yang-Baxter Equation*. Funct. Anal. Appl. **16** (1982) 27-34.

[37] E.K. Sklyanin, *Some Algebraic Structures Connected with the Yang-Baxter Equation: Representations of Quantum Algebras*. Funct. Anal. Appl. **17** (1983) 34-48.